Structural completeness for discriminator varieties

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Deductive systems

$Sent$ - set of propositional sentences
Ax - axioms ($\subseteq Sent$)

$+$ inference rules: $\dfrac{\Delta}{\varphi}$, $\Delta \subseteq_{fin} Sent$, $\varphi \in Sent$
(only structural rules - closed on substitutions)

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deductive system (Ax, R)

φ is a theorem of (Ax, R) provided

there is a proof of φ from Ax with the use of R

Shortening proofs

Can we shorten proofs of theorems (without adding new axioms)?

Yes, we can
by adding derivable rules

Can we do better?

Yes, we can
by adding admissible rules

Deductive systems, Structural Completeness

Associate with (Ax, R) a consequence relation

$$\vdash \subseteq \mathcal{P}(Sent) \times Sent$$

A rule $\dfrac{\Delta}{\varphi}$ is derivable if

$$\Delta \vdash \varphi$$

is admissible if for each substitution σ

$$(\forall \psi \in \Delta) \vdash \sigma(\psi) \quad \text{implies} \quad \vdash \sigma(\varphi)$$

Structural completeness

Examples of admissible non-derivable rules

$$\text{Harrop rule } \frac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)} \quad \text{in INT}$$

$$\frac{\Diamond x \wedge \Diamond \neg x}{\bot} \quad \text{in S5}$$

A deductive system is structurally complete (SC) if every its admissible rule is derivable

Examples for SC

classical logic, Gödel-Dummett logic, $\mathrm{INT}^{\to}$, Medvedev logic, S4.3Grz

Almost structural completeness

WAIT, there are admissible rules that cannot be used in proofs anyway

A rule $\dfrac{\Delta}{\varphi}$ is passive if

$$(\forall \sigma \in \mathit{Subst})(\exists \delta \in \Delta) \nvdash \sigma(\delta)$$

A deductive system is almost structurally complete (ASC) if its admissible non-derivable rules are passive

Examples for ASC\SC

S5, S4.3, $Ł_n$

SC vs ASC problem

How common is to be $ASC \setminus SC$

Maybe it is a negligible issue.

Problem

Determine which ASC deductive systems are SC.

Quasivarieties

Quasi-identities look like

$$(\forall \bar{x})\ s_1(\bar{x}) \approx t_1(\bar{t}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \rightarrow s(\bar{x}) \approx t(\bar{x})$$

Quasivarieties look like

Mod(quasi-identities)

Correspondence for algebraizable deductive systems

consequence relation $\vdash$	$\leftrightsquigarrow$	quasivariety $\mathcal{Q}$
deductive system (Ax, R)	$\leftrightsquigarrow$	axiomatization of $\mathcal{Q}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	identities true in $\mathcal{Q}$
derivable rules	$\leftrightsquigarrow$	quasi-identities true in $\mathcal{Q}$
admissible rules	$\leftrightsquigarrow$	quasi-identities true in $\mathbf{F}$

SC and ASC algebraically

F - $\mathcal{Q}$-algebra over $\aleph_0$ generators
Q(**F**) - quasivariety generated by **F**

A quasivariety $\mathcal{Q}$ is SC if $\mathcal{Q} = \mathrm{Q}(\mathbf{F})$, i.e., every quasi-identity valid in **F** is valid in $\mathcal{Q}$ too.

$\mathcal{Q}$ is ASC if for every quasi-identity q valid in **F** either q is valid in $\mathcal{Q}$ or its premises are not satisfiable in **F**,
i.e., every non-passive quasi-identity valid in **F** is also valid in $\mathcal{Q}$.

Theorem (W. Dzik, M.S.)

The following conditions are equivalent

- $\mathcal{Q}$ is ASC
- For every $\mathbf{A} \in \mathcal{Q}$, $(\exists h\colon \mathbf{A} \to \mathbf{F})$ yields $\mathbf{A} \in \mathrm{Q}(\mathbf{F})$

SC vs ASC problem algebraically

Problem

Determine which ASC quasivarieties are SC.

Partial solutions to SC vs ASC problem

Theorem (W. Dzik, M.S.)

$\mathcal{Q}$ is ASC iff it is SC provided

- $\mathbf{F}$ has an idempotent element (groups, lattices)
- every nontrivial algebra from $\mathcal{Q}$ admits a homomorphism into $\mathbf{F}$ (Heyting algebras, McKinsey algebras)

Theorem (W. Dzik, M.S.)

Let $\mathcal{V}$ be an ASC variety of closure algebras. The following conditions are equivalent

- $\mathcal{V}$ is SC
- $\mathcal{V}$ is a variety of McKinsey algebras
- There is no 4-element simple algebra in $\mathcal{V}$

General solution

Theorem (M.S.)

Let $\mathcal{Q}$ be an ASC quasivariety. The following conditions are equivalent

- $\mathcal{Q}$ is SC
- Every nontrivial $\mathcal{Q}$-finitely presented algebra admits a homomorphism into $\mathbf{F}$

Solution for semisimple quasivarieties

Theorem (M.S.)

Let $\mathcal{Q}$ be an ASC semisimple quasivariety. The following conditions are equivalent

- $\mathcal{Q}$ is SC
- $\mathcal{Q}$ is minimal or $\mathbf{F}$ has an elementary extension with an idempotent element

Discrimintor varieties

Theorem

Let $\mathcal{V}$ be a discriminator variety. Then

- $\mathcal{V}$ is semisimple
- $\mathcal{V}$ is ASC (S. Burris, W. Dzik)

Corollary

Let $\mathcal{V}$ be a discriminator variety. The following conditions are equivalent

- $\mathcal{V}$ is SC
- $\mathcal{V}$ is a minimal quasivariety or $\mathbf{F}$ has an elementary extension with an idempotent element

Discrimintor varieties in logic

Corollary

Let $\mathcal{V}$ be an SC discriminator variety. Assume that there are two distinct constants in $\mathbf{F}$ (like $\mathbf{0}$ and $\mathbf{1}$). Then $\mathcal{V}$ must be minimal.

Example

1. All (continuum many) varieties of two-dimensional cylindric algebras are ASC. There are only three such SC varieties: two minimal and one trivial
2. All (continuum many) varieties of relation algebras are ASC. There are only four such SC varieties: three minimal and one trivial

Minimal varieties

Problem
Let $\mathcal{V}$ be a minimal discriminator variety. Must $\mathcal{V}$ be minimal as a quasivariety?

Yes if $\mathcal{V}$ is finitely generated or the language of $\mathcal{V}$ is finite

But I constructed a huge counterexample which was wrong

Then I asked Miguel Campercholi and Diego Vaggione

Last Week Theorem (M. Campercholi, D. Vaggione)
Every minimal discriminator variety is minimal as a quasivariety.

Question

Problem

How many there are ASC$\setminus$SC varieties of closure algebras?

The end

This is all Thank you!